



# Generic controllability properties for the bilinear Schrödinger equation

Paolo Mason, Mario Sigalotti

## ► To cite this version:

Paolo Mason, Mario Sigalotti. Generic controllability properties for the bilinear Schrödinger equation. Communications in Partial Differential Equations, 2010, 35, pp.685-706. 10.1080/03605300903540919 . hal-00385279v2

**HAL Id: hal-00385279**

**<https://hal.science/hal-00385279v2>**

Submitted on 30 Nov 2009

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Generic controllability properties for the bilinear Schrödinger equation\*

Paolo Mason<sup>†</sup> Mario Sigalotti<sup>‡</sup>

November 30, 2009

## Abstract

In [15] we proposed a set of sufficient conditions for the approximate controllability of a discrete-spectrum bilinear Schrödinger equation. These conditions are expressed in terms of the controlled potential and of the eigenpairs of the uncontrolled Schrödinger operator. The aim of this paper is to show that these conditions are generic with respect to the uncontrolled and the controlled potential, denoted respectively by  $V$  and  $W$ . More precisely, we prove that the Schrödinger equation is approximately controllable generically with respect to  $W$  when  $V$  is fixed and also generically with respect to  $V$  when  $W$  is fixed and non-constant. The results are obtained by analytic perturbation arguments and through the study of asymptotic properties of eigenfunctions.

## 1 Introduction

In this paper we consider controlled Schrödinger equations of the type

$$i\frac{\partial\psi}{\partial t}(t, x) = (-\Delta + V(x) + u(t)W(x))\psi(t, x), \quad u(t) \in U, \quad (1)$$

where the *wave function*  $\psi$  is a map from  $[0, +\infty) \times \Omega$  to  $\mathbf{C}$  for some domain  $\Omega$  of  $\mathbf{R}^d$ ,  $d \geq 1$ ,  $V, W$  are real-valued functions and  $U$  is a nonempty subset of  $\mathbf{R}$ . We will assume either that  $\Omega, V, W$  are bounded and  $\psi|_{[0, +\infty) \times \partial\Omega} = 0$  or that  $\Omega = \mathbf{R}^d$  and  $-\Delta + V + uW$  has discrete spectrum for every  $u \in U$ .

As proved in [35], the control system (1) is never exactly controllable in  $L^2(\Omega)$ . Approximate controllability is known not to hold for some specific system of type (1) (see [23]). Nevertheless, several positive controllability results have been proved in recent years. Among them, let us mention [10, 12], where the exact controllability among regular enough wave

---

\*This work was supported by the BQR “Contrôle effectif des systèmes quantiques”, Région Lorraine – Nancy Université.

<sup>†</sup>P. Mason is with Laboratoire des Signaux et Systèmes, Supélec, 3, Rue Joliot Curie, 91192 Gif s/Yvette, France, [paolo.mason@lss.supelec.fr](mailto:paolo.mason@lss.supelec.fr)

<sup>‡</sup>M. Sigalotti is with INRIA Nancy - Grand Est, Équipe-projet CORIDA, and Institut Élie Cartan, UMR CNRS/INRIA/Nancy Université, BP 239, 54506 Vandœuvre-lès-Nancy, France, [Mario.Sigalotti@inria.fr](mailto:Mario.Sigalotti@inria.fr)

functions is proved for  $d = 1$ ,  $\Omega$  bounded and  $V = 0$ , and [25] for the  $L^2$ -approximate controllability when  $\Omega$  is bounded under suitable conditions on  $V$  and  $W$ . Other controllability results for related systems have been obtained in [1] (when more than one control is available) and in [22] (when the spectrum has only finitely many discrete eigenvalues).

In this paper we focus on the approximate controllability results obtained by the authors in collaboration with U. Boscain and T. Chambrion in [15]. Such results are related to those in [25] although the sets of sufficient conditions proposed in the two papers are incomparable and the techniques used are completely different: [25] applies a control Lyapunov function approach, whereas [15] is based on geometric control methods for the Galerkin approximations in the spirit of [3, 31]. As a consequence, the results in [15] are valid also in the case in which  $\Omega = \mathbf{R}^d$  (unlike those in [25]) and when  $\Omega$  is a manifold and  $\Delta$  is the Laplace-Beltrami operator. It should also be mentioned that the sufficient conditions for approximate controllability proposed in [15] imply stronger control properties such as control of density matrices (see Section 2) or tracking of unfeasible trajectories (see [14]). The aim of this paper is to show that such sufficient conditions are generic.

The genericity issue for the controllability of the Schrödinger equation has already been addressed in the literature. In particular, [25, Lemma 3.12] proves generic  $L^2$ -approximate controllability with respect to the pair  $(V, W)$  when  $d = 1$  and  $\Omega$  is bounded. Newer results can be found in [24]. Generic  $L^2$ -approximate controllability with respect to  $(\Omega, W)$  in the case  $V = 0$  is obtained in [26] as a consequence of generic properties of the Laplace-Dirichlet operator. Other generic controllability results for a linearized Schrödinger equation can be found in [11] and are further discussed in Section 5.

The difference between our approach and those usually adopted to prove generic properties of controlled partial differential equations is that, instead of applying local infinitesimal variations, we exploit global, long-range, perturbations. The idea is the following: denote by  $\Gamma$  the class of systems on which the genericity of a certain property  $P$  is studied. If we are able to prove the existence of at least one element of  $\Gamma$  satisfying  $P$ , then we can *propagate*  $P$  if some analytic dependence properties hold true. In this way we can prove that the property holds in a dense subset of  $\Gamma$ .

The paper is organized as follows: in Section 2 we describe the notion of solution of (1) (this is a delicate point when  $\Omega = \mathbf{R}^d$  and  $W$  is unbounded) and we recall the approximate controllability results obtained in [15] (Theorem 2.6). In particular, we formulate the two conditions ensuring approximate controllability: (i) the spectrum of  $-\Delta + V$  is non-resonant and (ii) the potential  $W$  couples, directly or indirectly, every pair of eigenvectors of  $-\Delta + V$ . We also recall the notion of genericity and we detail the topologies with respect to which genericity is considered. In Section 3 we prove the generic approximate controllability of (1) with respect to the pair  $(V, W)$ . As an intermediate step, we prove in Proposition 3.2 that, generically with respect to  $V$ , the spectrum of  $-\Delta + V$  is non-resonant. Section 4 refines the results of Section 3 by showing that the approximate controllability is generic separately with respect to  $V$  or  $W$  when  $(\Omega, W)$  or  $(\Omega, V)$  is fixed (in the first case, assuming that  $W$  is non-constant). We conclude the paper with Section 5, where we discuss the genericity with respect to  $\Omega$  of the approximate controllability of (1) when  $(V, W)$  is fixed.

## 2 Mathematical framework

### 2.1 Notations and definition of solutions

Let  $\mathbf{N}$  be the set of positive integers. For  $d \in \mathbf{N}$ , denote by  $\Xi_d$  the set of nonempty, open, bounded and connected subsets of  $\mathbf{R}^d$  and let  $\Xi_d^\infty = \Xi_d \cup \{\mathbf{R}^d\}$ . Take  $U \subset \mathbf{R}$  and assume that 0 belongs to  $U$ .

In the following we consider the Schrödinger equation (1) assuming that the potentials  $V, W$  are taken in  $L^\infty(\Omega, \mathbf{R})$  if  $\Omega$  belongs to  $\Xi_d$  and that  $V, W \in L_{\text{loc}}^\infty(\mathbf{R}^d, \mathbf{R})$  and  $\lim_{\|x\| \rightarrow \infty} V(x) + uW(x) = +\infty$  for every  $u \in U$  if  $\Omega = \mathbf{R}^d$ . Then, for every  $u \in U$ ,  $-\Delta + V + uW$  (with Dirichlet boundary conditions if  $\Omega$  is bounded) is a skew-adjoint operator on  $L^2(\Omega, \mathbf{C})$  with discrete spectrum (see [16, 27]). In particular,  $-\Delta + V + uW$  generates a group of unitary transformations  $e^{it(-\Delta+V+uW)} : L^2(\Omega, \mathbf{C}) \rightarrow L^2(\Omega, \mathbf{C})$ . Therefore,  $e^{it(-\Delta+V+uW)}(\mathcal{S}) = \mathcal{S}$  where  $\mathcal{S}$  denotes the unit sphere of  $L^2(\Omega, \mathbf{C})$ .

When  $\Omega$  is bounded, for every  $u \in L^\infty([0, T], U)$  and every  $\psi_0 \in L^2(\Omega, \mathbf{C})$  there exists a unique weak (and mild) solution  $\psi(\cdot; \psi_0, u) \in \mathcal{C}([0, T], L^2(\Omega, \mathbf{C}))$ . Moreover, if  $\psi_0 \in D(A)$  and  $u \in \mathcal{C}^1([0, T], U)$  then  $\psi(\cdot; \psi_0, u)$  is differentiable and it is a strong solution of (1). (See [9] and references therein.)

The situation is more complicated when  $\Omega = \mathbf{R}^d$  and  $W$  is unbounded. However, due to the well-definedness of  $e^{it(-\Delta+V+uW)}$  for every  $u \in U$  and  $t \in \mathbf{R}$ , we can always associate a solution

$$\psi(t; \psi_0, u) = e^{(t - \sum_{l=1}^{j-1} t_l)(-\Delta+V+u_j W)} \circ e^{t_{j-1}(-\Delta+V+u_{j-1} W)} \circ \dots \circ e^{t_1(-\Delta+V+u_1 W)}(\psi_0), \quad (2)$$

with every initial condition  $\psi_0 \in L^2(\Omega, \mathbf{R})$  and every piecewise constant control function  $u(\cdot)$ . Here  $\sum_{l=1}^{j-1} t_l \leq t < \sum_{l=1}^j t_l$  and

$$u(\tau) = u_k \quad \text{if } \sum_{l=1}^{k-1} t_l \leq \tau < \sum_{l=1}^k t_l$$

for  $k = 1, \dots, j$ .

**Definition 2.1.** We say that the quadruple  $(\Omega, V, W, U)$  is approximately controllable if for every  $\psi_0, \psi_1 \in \mathcal{S}$  and every  $\varepsilon > 0$  there exist  $T > 0$  and  $u : [0, T] \rightarrow U$  piecewise constant such that  $\|\psi_1 - \psi(T; \psi_0, u)\| < \varepsilon$ .

It is useful for the applications to extend the notion of approximate controllability from a single Schrödinger equation to a (possibly infinite) family of identical systems with different initial conditions, through the study of the evolution of the associated density matrix (see [5, 13]).

Let  $(\varphi_j)_{j \in \mathbf{N}}$  be an orthonormal basis of  $L^2(\Omega, \mathbf{C})$ ,  $(P_j)_{j \in \mathbf{N}}$  be a sequence of non-negative numbers such that  $\sum_{j=1}^\infty P_j = 1$ , and denote by  $\rho$  the *density matrix*

$$\rho = \sum_{j=1}^\infty P_j \varphi_j \varphi_j^*,$$

where  $\psi^*(\cdot) = \langle \psi, \cdot \rangle$ , for  $\psi \in L^2(\Omega, \mathbf{C})$  and  $\langle \cdot, \cdot \rangle$  denotes the scalar product in  $L^2$ . According to the classical definition of density matrix,  $\rho$  is a non-negative, self-adjoint operator of trace

class (see [28]). If each  $\varphi_j = \varphi_j(t)$  is interpreted as the state of a Schrödinger equation of the form (1), each equation being characterized by the same potentials  $V$  and  $W$  and driven by the same piecewise constant control  $u = u(t)$ , then the time evolution of the density matrix  $\rho = \rho(t)$  is described by

$$\rho(t) = \mathbf{U}(t)\rho(0)\mathbf{U}^*(t) = \sum_{j=1}^{\infty} P_j \mathbf{U}(t)\varphi_j(0)(\mathbf{U}(t)\varphi_j(0))^* \quad (3)$$

where the operator  $\mathbf{U}(t)$  is defined by

$$\mathbf{U}(t)\psi_0 = \psi(t; \psi_0, u) \quad (4)$$

and  $\mathbf{U}^*(t)$  denotes the adjoint of  $\mathbf{U}(t)$ .

**Definition 2.2.** *Two density matrices  $\rho_0$  and  $\rho_1$  are said to be unitarily equivalent if there exists a unitary transformation  $\mathbf{U}$  of  $L^2(\Omega, \mathbf{C})$  such that  $\rho_1 = \mathbf{U}\rho_0\mathbf{U}^*$ .*

For closed systems the problem of connecting two density matrices by a feasible trajectory makes sense only for pairs of density matrices that are unitarily equivalent. (The situation is different for open systems, see for instance [7].)

**Definition 2.3.** *We say that the quadruple  $(\Omega, V, W, U)$  is approximately controllable in the sense of density matrices if for every pair  $\rho_0, \rho_1$  of unitarily equivalent density matrices and every  $\varepsilon > 0$  there exist  $T > 0$  and  $u : [0, T] \rightarrow U$  piecewise constant such that  $\|\rho_1 - \mathbf{U}(T)\rho_0\mathbf{U}(T)^*\| < \varepsilon$ , where  $\|\cdot\|$  denotes the operator norm on  $L^2(\Omega, \mathbf{C})$  and  $\mathbf{U}$  is defined as in (4).*

It is clear that approximate controllability in the sense of density matrices implies approximate controllability (just take  $P_1 = 1$ ).

In order to state the approximate controllability result obtained in [15], we need to recall the following two definitions.

**Definition 2.4.** *The elements of a sequence  $(\mu_n)_{n \in \mathbf{N}} \subset \mathbf{R}$  are said to be  $\mathbf{Q}$ -linearly independent (equivalently, the sequence is said to be non-resonant) if for every  $K \in \mathbf{N}$  and  $(q_1, \dots, q_K) \in \mathbf{Q}^K \setminus \{0\}$  one has  $\sum_{n=1}^K q_n \mu_n \neq 0$ .*

**Definition 2.5.** *A  $n \times n$  matrix  $C = (c_{jk})_{1 \leq j, k \leq n}$  is said to be connected if for every pair of indices  $j, k \in \{1, \dots, n\}$  there exists a finite sequence  $r_1, \dots, r_l \in \{1, \dots, n\}$  such that  $c_{jr_1} c_{r_1 r_2} \cdots c_{r_{l-1} r_l} c_{r_l k} \neq 0$ .*

In the following we denote by  $\sigma(\Omega, V) = (\lambda_j(\Omega, V))_{j \in \mathbf{N}}$  the non-decreasing sequence of eigenvalues of  $-\Delta + V$ , counted according to their multiplicities, and by  $(\phi_j(\Omega, V))_{j \in \mathbf{N}}$  a corresponding sequence of eigenfunctions. Without loss of generality we can assume that  $\phi_j(\Omega, V)$  is real-valued for every  $j \in \mathbf{N}$ . Recall moreover that  $(\phi_j(\Omega, V))_{j \in \mathbf{N}}$  forms an orthonormal basis of  $L^2(\Omega, \mathbf{C})$ . If  $j \in \mathbf{N}$  is such that  $\lambda_j(\Omega, V)$  is simple, then  $\phi_j(\Omega, V)$  is uniquely defined up to sign.

## 2.2 Basic facts

The theorem below recalls the controllability results obtained in [15, Theorems 3.4, 5.2]. Here and in the following a map  $h : \mathbf{N} \rightarrow \mathbf{N}$  is called a *reordering* of  $\mathbf{N}$  if it is a bijection.

**Theorem 2.6.** *Let either (i)  $\Omega \in \Xi_d$ ,  $V, W \in L^\infty(\Omega, \mathbf{R})$  or (ii)  $\Omega = \mathbf{R}^d$ ,  $V, W \in L^\infty_{\text{loc}}(\mathbf{R}^d, \mathbf{R})$ ,  $\lim_{|x| \rightarrow \infty} V(x) + uW(x) = +\infty$  for every  $u \in U$ , and  $|W|$  have at most exponential growth at infinity. Assume that  $U$  contains the interval  $[0, \delta)$  for some  $\delta > 0$ , that the elements of  $(\lambda_{k+1}(\Omega, V) - \lambda_k(\Omega, V))_{k \in \mathbf{N}}$  are  $\mathbf{Q}$ -linearly independent and that there exists a reordering  $h : \mathbf{N} \rightarrow \mathbf{N}$  such that for infinitely many  $n \in \mathbf{N}$  the matrix*

$$B_n^h(\Omega, V, W) := \left( \int_{\Omega} W(x) \phi_{h(j)}(\Omega, V) \phi_{h(k)}(\Omega, V) dx \right)_{j,k=1}^n$$

*is connected (i.e.,  $B_n^h(\Omega, V, W)$  is frequently connected). Then  $(\Omega, V, W, U)$  is approximately controllable in the sense of density matrices.*

**Remark 2.7.** *Notice that, even in the unbounded case, each integral  $\int_{\Omega} W(x) \phi_j(\Omega, V) \phi_k(\Omega, V) dx$  is well defined. Indeed, when  $\Omega = \mathbf{R}^d$ , the growth of  $|W|$  is at most exponential and  $e^{a|x|} \phi_j(\mathbf{R}^d, V) \in L^2(\mathbf{R}^d, \mathbf{R})$  for every  $a > 0$  and  $j \in \mathbf{N}$  (see [2]).*

Let  $\mathcal{V}(\Omega)$  be equal to  $L^\infty(\Omega, \mathbf{R})$  if  $\Omega \in \Xi_d$  or to  $\{V \in L^\infty_{\text{loc}}(\mathbf{R}^d, \mathbf{R}) \mid \lim_{\|x\| \rightarrow \infty} V(x) = +\infty\}$  if  $\Omega = \mathbf{R}^d$ , and endow  $\mathcal{V}(\Omega)$  with the  $L^\infty$  topology.

Let us recall some useful perturbation results describing the dependence on  $V$  of the spectrum of the operator  $-\Delta + V$ .

**Theorem 2.8** (Continuity). *Let  $\Omega \in \Xi_d^\infty$ . Assume that  $\bar{V}$  belongs to  $\mathcal{V}(\Omega)$  and that  $\lambda_k(\Omega, \bar{V})$  is simple. Then there exists a neighborhood  $\mathcal{N}$  of  $\bar{V}$  in  $\mathcal{V}(\Omega)$  such that  $\lambda_k(\Omega, V)$  is simple for every  $V \in \mathcal{N}$  and  $V \mapsto \lambda_k(\Omega, V)$  depends continuously on  $V$  on  $\mathcal{N}$ . Moreover, the map  $V \mapsto \phi_k(\Omega, V)$  (defined up to the sign) is continuous from  $\mathcal{N}$  to  $L^2(\Omega, \mathbf{R})$ .*

The theorem follows from the remark that, if  $V$  tends to  $\bar{V}$  in  $\mathcal{V}(\Omega)$ , then the difference between the two operators  $-\Delta + V$  and  $-\Delta + \bar{V}$  tends to zero in norm. Therefore, the corresponding resolvents converge in norm, leading to the convergence of eigenvalues and spectral projections (see [18]).

We will need in the following a stronger continuity result.

**Proposition 2.9.** *Let  $\Omega = \mathbf{R}^d$ . Assume that  $\bar{V}$  belongs to  $\mathcal{V}(\mathbf{R}^d)$ ,  $\lambda_k(\mathbf{R}^d, \bar{V})$  is simple, and  $W \in L^\infty_{\text{loc}}(\mathbf{R}^d, \mathbf{R})$  be such that  $|W|$  has at most exponential growth. Then there exists a neighborhood  $\mathcal{N}$  of  $\bar{V}$  in  $\mathcal{V}(\mathbf{R}^d)$  such that  $\lambda_k(\mathbf{R}^d, V)$  is simple for every  $V \in \mathcal{N}$  and  $V \mapsto \sqrt{W} \phi_k(\mathbf{R}^d, V)$  (defined up to sign) is a continuous function from  $\mathcal{N}$  to  $L^2(\mathbf{R}^d, \mathbf{C})$ .*

*Proof.* Let  $\mathcal{N}$  be a neighborhood of  $\bar{V}$  such that  $\lambda_k(\mathbf{R}^d, V)$  is simple for every  $V \in \mathcal{N}$  (Theorem 2.8). Fix  $C, \alpha > 0$  such that  $|W(x)| < Ce^{\alpha|x|}$  almost everywhere on  $\mathbf{R}^d$ . Let, moreover,  $\alpha'$  be a constant larger than  $\alpha$ .

The estimates obtained in [2] (Theorems 4.1, 4.3 and 4.4) imply that, up to taking a smaller  $\mathcal{N}$ , there exists  $K > 0$  such that

$$\int_{\mathbf{R}^d} e^{\alpha'|x|} \phi_k(\mathbf{R}^d, V)^2 dx < K$$

for every  $V \in \mathcal{N}$ .

Let  $(V_n)_{n \in \mathbf{N}}$  be a sequence converging to  $\bar{V}$  in  $\mathcal{V}(\mathbf{R}^d)$ . Since  $\alpha' > \alpha$ , given  $\varepsilon > 0$ , there exists  $R > 0$  such that for every  $n$  large enough

$$\begin{aligned} \int_{\{x \in \mathbf{R}^d \mid \|x\| > R\}} |W(x)| (\phi_k(\mathbf{R}^d, V_n)^2 + \phi_k(\mathbf{R}^d, V)^2) dx < \\ \int_{\{x \in \mathbf{R}^d \mid \|x\| > R\}} C e^{\alpha|x|} (\phi_k(\mathbf{R}^d, V_n)^2 + \phi_k(\mathbf{R}^d, V)^2) dx < \varepsilon. \end{aligned}$$

Moreover, by continuity of  $V \mapsto \phi_k(\mathbf{R}^d, V)$  from  $\mathcal{N}$  to  $L^2(\mathbf{R}^d, \mathbf{R})$  and since  $W \in L^\infty_{\text{loc}}(\mathbf{R}^d, \mathbf{R})$ ,

$$\int_{\{x \in \mathbf{R}^d \mid \|x\| \leq R\}} |W(x)| (\phi_k(\mathbf{R}^d, V_n) - \phi_k(\mathbf{R}^d, V))^2 dx < \varepsilon$$

for every  $n$  large enough. Therefore,  $\sqrt{W} \phi_k(\mathbf{R}^d, V_n)$  converges to  $\sqrt{W} \phi_k(\mathbf{R}^d, V)$  in  $L^2(\mathbf{R}^d, \mathbf{C})$  as  $n$  tends to infinity.  $\square$

Another crucial result for our needs concerns analytic perturbation properties.

**Theorem 2.10** ([18, Chapter VII], [29, Chapter II]). *Let  $I$  be an interval of  $\mathbf{R}$  and  $\Omega$  belong to  $\Xi_d^\infty$ . Assume that  $V$  belongs to  $\mathcal{V}(\Omega)$  and that  $\mu \mapsto W_\mu$  is an analytic function from  $I$  into  $L^\infty(\Omega, \mathbf{R})$ . Then, there exist two families of analytic functions  $(\Lambda_k : I \rightarrow \mathbf{R})_{k \in \mathbf{N}}$  and  $(\Phi_k : I \rightarrow L^2(\Omega, \mathbf{R}))_{k \in \mathbf{N}}$  such that for any  $\mu$  in  $I$  the sequence  $(\Lambda_k(\mu))_{k \in \mathbf{N}}$  is the family of eigenvalues of  $-\Delta + V + W_\mu$  counted according to their multiplicities and  $(\Phi_k(\mu))_{k \in \mathbf{N}}$  is an orthonormal basis of corresponding eigenfunctions.*

In the following sections we will also make use of the stronger analytic dependence result stated below.

**Proposition 2.11.** *Let  $\Omega$  belong to  $\Xi_d^\infty$  and  $\{V_\mu \mid \mu \in [0, 1]\}$  be a family of functions in  $\mathcal{V}(\Omega)$  such that  $V_\mu - V_0$  is analytic in  $L^\infty(\Omega)$  with respect to  $\mu$ . Let  $W \in L^\infty_{\text{loc}}(\Omega, \mathbf{R})$  be such that  $|W(x)| \leq C(|V_0(x)| + 1)$  for almost every  $x \in \Omega$ , for some positive constant  $C$ . Then, if the eigenvalues  $\lambda_j(\Omega, V_\mu)$  and  $\lambda_k(\Omega, V_\mu)$  are simple for  $\mu \in (0, 1)$ , the map*

$$\mu \mapsto \int_{\Omega} W \phi_j(\Omega, V_\mu) \phi_k(\Omega, V_\mu)$$

*is analytic from  $(0, 1)$  to  $\mathbf{R}$ .*

*Proof.* First of all notice that, when  $\Omega$  is bounded, the proposition follows directly from Theorem 2.10. Let then  $\Omega = \mathbf{R}^d$ . Since the scalar product in  $L^2(\mathbf{R}^d, \mathbf{C})$  is analytic, it is enough to prove that the map  $\mu \mapsto \sqrt{W} \phi_k(\mathbf{R}^d, V_\mu)$  is analytic in  $L^2(\mathbf{R}^d, \mathbf{C})$  if  $\lambda_k(\mathbf{R}^d, V_\mu)$  is simple for  $\mu \in (0, 1)$ .

Let us first show that, setting  $T_\mu = -\Delta + V_\mu$  and endowing  $D(T_0)$  with the graph norm  $\|\phi\|_{T_0} = \|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} + \|T_0 \phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}$ , the eigenfunctions  $\phi_k(\mathbf{R}^d, V_\mu)$  are analytic from  $(0, 1)$  to  $D(T_0)$ . (This is essentially done in [30, Theorem 5.6]. We adapt the argument to our case for the sake of completeness.) Take  $\lambda_0$  in the resolvent set of the operator  $T_{\mu_0}$ , for a fixed  $\mu_0 \in (0, 1)$ . For  $\mu$  in a neighborhood of  $\mu_0$  we have

$$(T_\mu - \lambda_0)^{-1} = (T_{\mu_0} - \lambda_0)^{-1} (\text{Id} + (V_\mu - V_{\mu_0}) (T_{\mu_0} - \lambda_0)^{-1})^{-1}, \quad (5)$$



where  $\text{Id}$  denotes the identity.

Note that  $\mu \mapsto (\text{Id} + (V_\mu - V_{\mu_0})(T_{\mu_0} - \lambda_0)^{-1})^{-1}$  is analytic in  $\mathcal{L}(L^2(\mathbf{R}^d, \mathbf{C}))$ , the space of linear and continuous operators of  $L^2(\mathbf{R}^d, \mathbf{C})$ , for  $\mu$  in a neighborhood of  $\mu_0$ .

Notice also that  $(T_{\mu_0} - \lambda_0)^{-1}$  belongs to  $\mathcal{L}(L^2(\mathbf{R}^d, \mathbf{C}), D(T_0))$ , the space of linear and continuous maps from  $L^2(\mathbf{R}^d, \mathbf{C})$  to  $D(T_0)$  (endowed with the graph norm), as it follows from the following series of inequalities:

$$\begin{aligned} \|(T_{\mu_0} - \lambda_0)^{-1}\phi\|_{T_0} &= \|(T_{\mu_0} - \lambda_0)^{-1}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} + \|T_0(T_{\mu_0} - \lambda_0)^{-1}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} \\ &\leq \|(T_{\mu_0} - \lambda_0)^{-1}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} + \|\phi + (\lambda_0 + V_0 - V_{\mu_0})(T_{\mu_0} - \lambda_0)^{-1}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} \\ &\leq (\|(T_{\mu_0} - \lambda_0)^{-1}\| + 1 + \|\lambda_0 + V_0 - V_{\mu_0}\|_{L^\infty(\mathbf{R}^d, \mathbf{C})}\|(T_{\mu_0} - \lambda_0)^{-1}\|)\|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}. \end{aligned}$$

Hence  $F \mapsto (T_{\mu_0} - \lambda_0)^{-1}F$  is a linear and continuous operator from  $\mathcal{L}(L^2(\mathbf{R}^d, \mathbf{C}))$  to  $\mathcal{L}(L^2(\mathbf{R}^d, \mathbf{C}), D(T_0))$ . It follows from (5) that  $\mu \mapsto (T_\mu - \lambda_0)^{-1}$  is analytic from a neighborhood of  $\mu_0$  to  $\mathcal{L}(L^2(\mathbf{R}^d, \mathbf{C}), D(T_0))$ .

Then the eigenfunction  $\phi_k(\mathbf{R}^d, V_\mu)$  is analytic with respect to  $\mu$  in  $D(T_0)$  since the spectral projection

$$(2\pi i)^{-1} \oint_{|\lambda - \lambda_k(\mathbf{R}^d, V_{\mu_0})| = \varepsilon} (T_\mu - \lambda)^{-1} d\lambda,$$

where  $\varepsilon$  is small enough, is analytic as a function of  $\mu$  taking values in  $\mathcal{L}(L^2(\mathbf{R}^d, \mathbf{C}), D(T_0))$ . (See [27, Theorem XII.8] for details.)

To conclude the proof of the proposition it is enough to check that the linear map from  $D(T_0)$  to  $L^2(\mathbf{R}^d, \mathbf{C})$  sending  $\phi$  to  $\sqrt{W}\phi$  is continuous, i.e., that there exists  $\hat{C} > 0$  such that  $\|\sqrt{W}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} \leq \hat{C}\|\phi\|_{T_0}$  for every  $\phi \in D(T_0)$ . We have

$$\begin{aligned} \|\sqrt{W}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 &\leq C \left( \|\sqrt{V_0}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 + \|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 \right) \\ &\leq C \left( \|\nabla\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 + \|\sqrt{V_0}\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 + \|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 \right) \\ &= C \left( \langle (-\Delta + |V_0|)\phi, \phi \rangle_{L^2(\mathbf{R}^d, \mathbf{C})} + \|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 \right) \\ &\leq C \left( \|\phi\|_{T_0} \|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})} + 2 \max\{0, -\text{ess inf } V_0\} \|\phi\|_{L^2(\mathbf{R}^d, \mathbf{C})}^2 \right) \\ &\leq \hat{C} \|\phi\|_{T_0}^2, \end{aligned}$$

where we can take  $\hat{C} = C(1 + 2 \max\{0, -\text{ess inf } V_0\})$ .  $\square$

The following proposition states the existence of analytic paths of potentials such that the spectrum is simple along them.

**Proposition 2.12.** *Let  $\Omega$  belong to  $\Xi_d^\infty$  and  $V, Z \in \mathcal{V}(\Omega)$  be such that  $Z - V \in L^\infty(\Omega, \mathbf{R})$ . Then there exists an analytic function  $\mu \mapsto W_\mu$  from  $[0, 1]$  into  $L^\infty(\Omega, \mathbf{R})$  such that  $W_0 = 0$ ,  $W_1 = Z - V$  and the spectrum of  $-\Delta + V + W_\mu$  is simple for every  $\mu \in (0, 1)$ .*

*Proof.* Denote by  $\mathcal{C}_0(\Omega)$  the subspace of  $L^\infty(\Omega)$  of continuous real-valued functions vanishing at infinity. Note that  $\mathcal{C}_0(\Omega)$  is a separable Banach space. The proof of the proposition is based on [33, Theorem B], which guarantees that the thesis holds true with  $W_\mu \in \mathcal{C}_0(\Omega) + \mathbf{R}(Z - V)$  provided that, for every  $W \in \mathcal{C}_0(\Omega) + \mathbf{R}(Z - V)$  and every multiple eigenvalue  $\lambda$  of  $-\Delta + V + W$ ,



there exist two orthonormal eigenfunctions  $\phi_1$  and  $\phi_2$  pertaining to  $\lambda$  such that the linear functionals

$$p \mapsto \int_{\Omega} p (\phi_1^2 - \phi_2^2) dx$$

and

$$p \mapsto \int_{\Omega} p \phi_1 \phi_2 dx$$

are linearly independent on  $\mathcal{C}_0(\Omega) + \mathbf{R}(Z - V)$ .

The linear independence of the two functionals can be proved by taking any pair of orthonormal eigenfunctions  $\phi_1$  and  $\phi_2$  pertaining to  $\lambda$  and assuming by contradiction that there exists  $(a_1, a_2) \in \mathbf{R}^2 \setminus \{(0, 0)\}$  such that

$$p \mapsto \int_{\Omega} p(a_1\phi_1^2 - a_1\phi_2^2 + a_2\phi_1\phi_2) dx$$

is identically equal to zero on  $\mathcal{C}_0(\Omega) + \mathbf{R}(Z - V)$ . Hence,  $a_1\phi_1^2 - a_1\phi_2^2 + a_2\phi_1\phi_2$  must be identically equal to zero on  $\Omega$ .

By diagonalizing the quadratic form  $a_1v_1^2 - a_1v_2^2 + a_2v_1v_2$  we end up with  $c_1, c_2 \in \{-1, 0, 1\}$  and two linearly independent eigenfunctions  $\psi_1$  and  $\psi_2$  pertaining to  $\lambda$  such that  $c_1^2 + c_2^2 > 0$  and

$$c_1\psi_1^2 + c_2\psi_2^2 \equiv 0.$$

Then for every  $x \in \Omega$  either  $\psi_1(x) = \psi_2(x)$  or  $\psi_1(x) = -\psi_2(x)$ . Thanks to the unique continuation property (see [32]) we have  $\psi_1 = \pm\psi_2$ , contradicting the linear independence of  $\psi_1$  and  $\psi_2$ .  $\square$

### 2.3 Genericity: topologies and definitions

From now on we will write simply  $L^p(\Omega)$  or  $L_{\text{loc}}^p(\Omega)$  to denote  $L^p(\Omega, \mathbf{R})$  or  $L_{\text{loc}}^p(\Omega, \mathbf{R})$  respectively. Similarly,  $H^p(\Omega)$  and  $H_0^p(\Omega)$  will denote  $H^p(\Omega, \mathbf{R})$  and  $H_0^p(\Omega, \mathbf{R})$ .

For every  $\Omega \in \Xi_d^\infty$  let  $\mathcal{W}(\Omega)$  be equal to  $L^\infty(\Omega)$  if  $\Omega \in \Xi_d$  or to

$$\{W \in L_{\text{loc}}^\infty(\mathbf{R}^d) \mid \text{ess sup}_{x \in \mathbf{R}^d} \frac{\log(|W(x)| + 1)}{\|x\| + 1} < \infty\}$$

if  $\Omega = \mathbf{R}^d$ . In both cases endow  $\mathcal{W}(\Omega)$  with the  $L^\infty$  topology. Denote

$$\mathcal{Z}(\Omega, U) = \{(V, W) \mid V \in \mathcal{V}(\Omega), W \in \mathcal{W}(\Omega), V + uW \in \mathcal{V}(\Omega) \text{ for every } u \in U\}$$

and endow  $\mathcal{Z}(\Omega, U)$  with the product  $L^\infty$  topology. We also introduce, for every  $V \in \mathcal{V}(\Omega)$  and every  $W \in \mathcal{W}(\Omega)$ , the topological subspaces of  $\mathcal{V}(\Omega)$  and  $\mathcal{W}(\Omega)$  defined, with a slight abuse of notation, by

$$\begin{aligned} \mathcal{V}(\Omega, W, U) &= \{\tilde{V} \in \mathcal{V}(\Omega) \mid (\tilde{V}, W) \in \mathcal{Z}(\Omega, U)\}, \\ \mathcal{W}(\Omega, V, U) &= \{\tilde{W} \in \mathcal{W}(\Omega) \mid (V, \tilde{W}) \in \mathcal{Z}(\Omega, U)\}. \end{aligned}$$

Notice that neither  $\mathcal{V}(\Omega, W, U)$  nor  $\mathcal{W}(\Omega, V, U)$  is empty. Moreover, both  $\mathcal{V}(\Omega, W, U)$  and  $\mathcal{W}(\Omega, V, U)$  are invariant by the set addition with  $L^\infty(\Omega)$ . In particular, they are open in  $\mathcal{V}(\Omega)$  and  $\mathcal{W}(\Omega)$  respectively and they coincide with  $L^\infty(\Omega)$  when  $\Omega \in \Xi_d$ .

Theorem 2.6 motivates the following definition.

**Definition 2.13.** Let  $V \in \mathcal{V}(\Omega)$  and  $W \in \mathcal{W}(\Omega)$ . We say that  $(\Omega, V, W)$  is fit for control if  $(\lambda_{k+1}(\Omega, V) - \lambda_k(\Omega, V))_{k \in \mathbf{N}}$  is non-resonant and  $B_n^h(\Omega, V, W)$  is frequently connected for some reordering  $h$ . Let  $(V, W)$  be an element of  $\mathcal{Z}(\Omega, U)$ . We say that the quadruple  $(\Omega, V, W, U)$  is effective if  $(\Omega, V + uW, W)$  is fit for control for some  $u$  such that  $[u, u + \delta) \subset U$  for some  $\delta > 0$ .

Theorem 2.6 can then be rephrased by saying that being effective is a sufficient condition for controllability in the sense of the density matrices. The rest of the paper deals with the genericity of the notions introduced in Definition 2.13.

Let us recall that a topological space  $X$  is called a *Baire space* if the intersection of countably many open and dense subsets of  $X$  is dense in  $X$ . Every complete metric space is a Baire space. The intersection of countably many open and dense subsets of a Baire space is called a *residual subset* of  $X$ . Given a Baire space  $X$ , a boolean function  $P : X \rightarrow \{0, 1\}$  is said to be a *generic property* if there exists a residual subset  $Y$  of  $X$  such that every  $x$  in  $Y$  satisfies property  $P$ , that is,  $P(x) = 1$ .

### 3 The triple $(\Omega, V, W)$ is generically fit for control with respect to the pair $(V, W)$

Let us start by recalling a known result on the generic simplicity of eigenvalues for Schrödinger operators on bounded domains (see [4]).

**Proposition 3.1** (Albert). Let  $\Omega$  belong to  $\Xi_d$ . For every  $k \in \mathbf{N}$  the set

$$\{V \in L^\infty(\Omega) \mid \lambda_1(\Omega, V), \dots, \lambda_k(\Omega, V) \text{ are simple}\}$$

is open and dense in  $L^\infty(\Omega)$ . Hence, the spectrum  $\sigma(\Omega, V)$  is simple generically with respect to  $V$ .

For every  $\Omega \in \Xi_d^\infty$  and every  $k \in \mathbf{N}$ , let

$$\mathcal{R}_k(\Omega) = \{V \in \mathcal{V}(\Omega) \mid \lambda_1(\Omega, V), \dots, \lambda_k(\Omega, V) \text{ are simple}\}. \quad (6)$$

We generalize Proposition 3.1 as follows.

**Proposition 3.2.** Let  $\Omega$  belong to  $\Xi_d^\infty$ . For every  $K \in \mathbf{N}$  and  $q = (q_1, \dots, q_K) \in \mathbf{Q}^K \setminus \{0\}$ , the set

$$\mathcal{O}_q(\Omega) = \left\{ V \in \mathcal{R}_K(\Omega) \mid \sum_{j=1}^K q_j \lambda_j(\Omega, V) \neq 0 \right\} \quad (7)$$

is open and dense in  $\mathcal{V}(\Omega)$ . Hence, the spectrum  $\sigma(\Omega, V)$  forms a non-resonant family generically with respect to  $V$ .

The proof of Proposition 3.2 is based on the following lemma.

**Lemma 3.3.** Let  $\Omega$  belong to  $\Xi_d^\infty$  and  $\omega$  be a nonempty, open subset compactly contained in  $\Omega$  and whose boundary is Lipschitz continuous. Let  $v \in L^\infty(\omega)$  and  $(V_k)_{k \in \mathbf{N}}$  be a sequence in  $\mathcal{V}(\Omega)$  such that  $V_k|_\omega \rightarrow v$  in  $L^\infty(\omega)$  as  $k \rightarrow \infty$  and  $\lim_{k \rightarrow \infty} \inf_{\Omega \setminus \omega} V_k = +\infty$ . Then, for every  $j \in \mathbf{N}$ ,  $\lim_{k \rightarrow \infty} \lambda_j(\Omega, V_k) = \lambda_j(\omega, v)$ . Moreover, if  $\lambda_j(\omega, v)$  is simple then (up to the sign)  $\phi_j(\Omega, V_k)$  and  $\sqrt{V_k} \phi_j(\Omega, V_k)$  converge respectively to  $\phi_j(\omega, v)$  and  $\sqrt{v} \phi_j(\omega, v)$  in  $L^2(\Omega, \mathbf{C})$  as  $k$  goes to infinity, where  $\phi_j(\omega, v)$  is identified with its extension by zero outside  $\omega$ .

*Proof of Lemma 3.3.* Without loss of generality we can assume that  $v \geq 1$  on  $\omega$  and  $V_k \geq 1$  on  $\Omega$ . Indeed, for  $k$  large enough  $V_k \geq 1$  on  $\Omega \setminus \omega$  and, for what concerns the values of  $V_k$  and  $v$  on  $\omega$ , it suffices to notice that if we replace  $v$  by  $v + c$  and each  $V_k$  by  $V_k + c$  with

$$c = \max\{\|v\|_{L^\infty(\omega)}, \|V_k\|_{L^\infty(\omega)} \mid k \in \mathbf{N}\} + 1,$$

then we simply operate a shift of the spectra  $\sigma(v, \omega)$  and  $\sigma(V_k, \Omega)$  by the common constant  $c$ . Therefore, the operators  $-\Delta + v$  and  $-\Delta + V_k$  are invertible and their inverses, denoted respectively by  $a : L^2(\omega) \rightarrow L^2(\omega)$  and  $A_k : L^2(\Omega) \rightarrow L^2(\Omega)$ , are compact and have uniformly bounded norm.

Let  $A : L^2(\Omega) \rightarrow L^2(\Omega)$  be the operator associating with  $f \in L^2(\Omega)$  the extension by zero on  $\Omega \setminus \omega$  of  $a(f|_\omega)$ . Let us prove that  $A_k$  converges pointwise to  $A$ .

Fix  $f \in L^2(\Omega)$  and denote, for every  $k \in \mathbf{N}$ ,  $w_k = A_k f$  and  $w = Af$ . Then  $w_k \in H_0^1(\Omega)$  is the weak solution of

$$(-\Delta + V_k)w_k = f \quad \text{in } \Omega$$

and  $w|_\omega \in H_0^1(\omega)$  is the weak solution of

$$(-\Delta + v)w = f \quad \text{in } \omega. \tag{8}$$

We must prove that  $\|w_k - w\|_{L^2(\Omega)}$  tends to zero as  $k$  goes to infinity.

By definition of weak solution, for every  $\varphi \in H_0^1(\Omega)$ ,

$$\int_{\Omega} \nabla w_k \cdot \nabla \varphi + \int_{\Omega} V_k w_k \varphi = \int_{\Omega} f \varphi \tag{9}$$

and, similarly, for every  $\psi \in H_0^1(\omega)$ ,

$$\int_{\omega} \nabla w \cdot \nabla \psi + \int_{\omega} v w \psi = \int_{\omega} f \psi.$$

Taking  $w_k$  as  $\varphi$  in (9) we easily get that the sequence  $w_k$  is uniformly bounded in  $H_0^1(\Omega)$ . Denote by  $w^*$  the limit of a subsequence of  $w_k$  weakly converging in  $H_0^1(\Omega)$  (whose existence is guaranteed by Banach-Alaoglu theorem). With a slight abuse of notation, let us identify  $w_k$  with its weakly converging subsequence.

The definition of weak convergence and (9) imply that, for every  $\psi \in H_0^1(\omega)$ ,

$$\int_{\omega} \nabla w^* \cdot \nabla \psi + \int_{\omega} v w^* \psi = \int_{\omega} f \psi.$$

Taking again  $\varphi = w_k$  in (9), we notice that  $\{\sqrt{\inf_{\Omega \setminus \omega} V_k} \|w_k\|_{L^2(\Omega \setminus \omega)}\}_{k \in \mathbf{N}}$  is a bounded sequence in  $\mathbf{R}$ . Hence,  $w_k \rightarrow 0$  in  $L^2(\Omega \setminus \omega)$  and thus  $w^* = 0$  on  $\Omega \setminus \omega$ . Recall that, since the boundary of  $\omega$  is Lipschitz continuous, then any  $H^1$  function which is defined in a neighborhood of  $\omega$  and which annihilates outside  $\omega$  belongs to  $H_0^1(\omega)$  (see [17, Lemma 3.2.15]). Hence  $w^* \in H_0^1(\omega)$  coincides with  $w$ , the unique weak solution of (8). Since a  $H^1$ -weakly converging sequence converges  $L^2$ -strongly on bounded sets (see, for instance, [19, Theorem 8.6]), then we have proved the convergence of  $w_k$  to  $w$  in  $L^2(\Omega)$ , that is, the pointwise convergence of  $A_k$  to  $A$ .

We claim that the family of operators  $\{A_k \mid k \in \mathbf{N}\}$  is collectively compact. Recall that  $\{A_k \mid k \in \mathbf{N}\}$  is collectively compact if for every  $(f_k)_{k \in \mathbf{N}}$  in the unit ball of  $L^2(\Omega)$ , the set

$\{A_k f_k \mid k \in \mathbf{N}\}$  is pre-compact (see [8]). The proof of this fact is quite classical and the argument proposed here follows a similar one given in [6, Lemma 5.1]. Let  $z_k = A_k f_k$  and notice that

$$\|\nabla z_k\|_{L^2(\Omega)}^2 + \|\sqrt{V_k} z_k\|_{L^2(\Omega)}^2 = \int_{\Omega} f_k z_k. \quad (10)$$

Since the  $A_k$ 's are uniformly bounded, then the right-hand side of (10) is uniformly bounded. Thus, up to extracting a subsequence,  $z_k$  weakly converges to some  $z$  in  $H_0^1(\Omega)$ . If  $\Omega \in \Xi_d$  then  $z_k \rightarrow z$  strongly in  $L^2(\Omega)$  which proves the collective compactness of  $A_k$  in this case. When  $\Omega = \mathbf{R}^d$  let  $\bar{V}(x) = \inf_{k \in \mathbf{N}} V_k(x)$  for every  $x \in \mathbf{R}^d$ . Then  $\bar{V}$  belongs to  $L_{\text{loc}}^\infty(\mathbf{R}^d)$ ,  $\bar{V} \geq 1$  almost everywhere and  $\lim_{\|x\| \rightarrow \infty} \bar{V}(x) = +\infty$ . In order to prove this last property, assume by contradiction that there exists a sequence  $x_k$  such that  $\lim_{k \rightarrow \infty} \|x_k\| = \infty$  and  $\bar{V}(x_k)$  is uniformly bounded. Then there exists a subsequence  $x_{k_j}$  such that either  $v_j = V_{m_j}(x_{k_j})$  is uniformly bounded for some  $m \in \mathbf{N}$  or  $V_{m_j}(x_{k_j})$  is uniformly bounded for some unbounded sequence  $(m_j)_{j \in \mathbf{N}}$  in  $\mathbf{N}$ . The contradiction follows in the first case from the fact that  $V_m \in \mathcal{V}(\mathbf{R}^d)$ , while in the second case it is a consequence of the convergence of  $\inf_{\mathbf{R}^d \setminus \omega} V_k$  to infinity as  $k$  goes to infinity. Then, for every  $\rho > 0$ ,

$$\|z_k - z\|_{L^2(\mathbf{R}^d)}^2 \leq \int_{\{\bar{V} < \rho\}} (z_k - z)^2 + \frac{1}{\rho} \int_{\{\bar{V} \geq \rho\}} (z_k - z)^2 \bar{V} \quad (11)$$

and  $\{\bar{V} < \rho\}$  is bounded. It follows from (10) that the  $L^2$ -norm of  $z_k \sqrt{\bar{V}}$  on  $\mathbf{R}^d$  is uniformly bounded with respect to  $k$ . Since  $z_k$  converges  $H^1$ -weakly to  $z$  in  $\mathbf{R}^d$ , and therefore  $L^2$ -strongly on each compact set, then  $z \sqrt{\bar{V}}$  belongs to  $L^2(\mathbf{R}^d)$ . We deduce that  $\int_{\{\bar{V} \geq \rho\}} (z_k - z)^2 \bar{V}$  is uniformly bounded with respect to  $k$  and thus, for  $\rho$  large enough,  $(1/\rho) \int_{\{\bar{V} \geq \rho\}} (z_k - z)^2 \bar{V}$  is arbitrarily small, uniformly with respect to  $k$ . Since  $\{\bar{V} < \rho\}$  is bounded, then, for any fixed  $\rho$ ,  $z_k \rightarrow z$  in  $L^2(\{\bar{V} < \rho\})$ . It follows from (11) that  $z_k$  converges to  $z$  in  $L^2(\mathbf{R}^d)$ , concluding the proof of the collective compactness of  $\{A_k\}_{k \in \mathbf{N}}$ .

Theorems 4.8 and 4.11 in [8] guarantee that  $\lambda_j(\Omega, V_k)$  converges to  $\lambda_j(\omega, v)$  as  $k$  goes to infinity for every  $j \in \mathbf{N}$  and that, if  $\lambda_j(\omega, v)$  is simple, then (up to the sign)  $\lim_{k \rightarrow \infty} \phi_j(\Omega, V_k) = \phi_j(\omega, v)$  in  $L^2(\Omega)$ .

To conclude the proof we observe that it is enough to show that the restriction of  $\sqrt{V_k} \phi_j(\Omega, V_k)$  to  $\Omega \setminus \omega$  converges to zero in  $L^2(\Omega \setminus \omega)$  as  $k$  goes to infinity, where  $j$  is such that  $\lambda_j(\omega, v)$  is simple.

Since  $\phi_j(\Omega, V_k)$  satisfies

$$(-\Delta + V_k) \phi_j(\Omega, V_k) = \lambda_j(\Omega, V_k) \phi_j(\Omega, V_k)$$

in the weak sense, then taking  $\phi_j(\Omega, V_k)$  as test function we have

$$\int_{\Omega} \|\nabla \phi_j(\Omega, V_k)\|^2 + \int_{\Omega} V_k \phi_j(\Omega, V_k)^2 = \lambda_j(\Omega, V_k). \quad (12)$$

In particular the sequence  $(\phi_j(\Omega, V_k))_{k \in \mathbf{N}}$  is uniformly bounded in  $H_0^1(\Omega)$  and so  $\phi_j(\Omega, V_k)$  converges to  $\phi_j(\omega, v)$  not only strongly in  $L^2(\Omega)$  but also weakly in  $H_0^1(\Omega)$ . Hence,

$$\begin{aligned} \limsup_{k \rightarrow \infty} \|\sqrt{V_k} \phi_j(\Omega, V_k)\|_{L^2(\Omega \setminus \omega)}^2 &= \lambda_j(\omega, v) - \int_{\omega} v \phi_j(\omega, v)^2 - \liminf_{k \rightarrow \infty} \|\nabla \phi_j(\Omega, V_k)\|_{L^2(\Omega)}^2 \\ &\leq \lambda_j(\omega, v) - \int_{\omega} v \phi_j(\omega, v)^2 - \|\nabla \phi_j(\omega, v)\|_{L^2(\Omega)}^2. \end{aligned}$$

The last term of the above inequality is equal to zero, as it follows from the analogous of (12) for  $\phi_j(\omega, v)$ .  $\square$

*Proof of Proposition 3.2.* The second part of the statement clearly follows from the first one, since

$$\{V \in \mathcal{V}(\Omega) \mid \sigma(\Omega, V) \text{ non-resonant}\} = \bigcap_{q \in \bigcup_{K \in \mathbf{N}} (\mathbf{Q}^K \setminus \{0\})} \mathcal{O}_q(\Omega).$$

For each  $K \in \mathbf{N}$  and  $q = (q_1, \dots, q_K) \in \mathbf{Q}^K \setminus \{0\}$ , the openness of  $\mathcal{O}_q(\Omega)$  in  $\mathcal{V}(\Omega)$  follows directly from the continuity of the eigenvalues on  $V$ . (See Theorem 2.8.)

We prove the density of  $\mathcal{O}_q(\Omega)$  in  $\mathcal{V}(\Omega)$  by an analytic perturbation argument. Fix  $V \in \mathcal{V}(\Omega)$ . Let  $\omega$  be a  $d$ -orthotope compactly contained in  $\Omega$  and  $v$  a measurable bounded function on  $\omega$  such that  $\sigma(\omega, v)$  is non-resonant. (The existence of such  $\omega$  and  $v$  is obtained in [15, Section 6.3] for  $d = 3$  and the proof extends with no extra difficulty to the general case  $d \in \mathbf{N}$ .)

Let us consider a sequence  $(V_k)_{k \in \mathbf{N}} \in \mathcal{V}(\Omega)$  such that  $V_k - V \in L^\infty(\Omega)$ ,  $\forall k \in \mathbf{N}$  and such that  $V_k|_\omega \rightarrow v$  in  $L^\infty(\omega)$  as  $k \rightarrow \infty$  and  $\lim_{k \rightarrow \infty} \inf_{\Omega \setminus \omega} V_k = +\infty$ . By Lemma 3.3 we have that  $\lim_{k \rightarrow \infty} \sum_{j=1}^K q_j \lambda_j(\Omega, V_k) = \sum_{j=1}^K q_j \lambda_j(\omega, v) \neq 0$  so that  $V_{\bar{k}} \in \mathcal{O}_q$  for some  $\bar{k}$  large enough.

By Proposition 2.12 we can construct an analytic path  $\mu \mapsto W_\mu$  from  $[0, 1]$  into  $L^\infty(\Omega)$  such that  $W_0 = 0$ ,  $W_1 = V_{\bar{k}} - V$  and the spectrum of  $-\Delta + V + W_\mu$  is simple for every  $\mu \in (0, 1)$ . This, together with Theorem 2.10, implies that the map  $\mu \mapsto \sum_{j=1}^K q_j \lambda_j(\Omega, V + W_\mu)$ , which is different from zero at  $\mu = 1$ , is analytic and thus different from zero almost everywhere. Hence,  $\mathcal{O}_q$  is dense in  $\mathcal{V}(\Omega)$ .  $\square$

The following theorem extends the analysis from  $V$  to the pair  $(V, W)$ , combining the generic non-resonance of the spectrum of  $-\Delta + V$  with the genericity of the connectedness of the matrices  $B_n^h(\Omega, V, W)$ .

**Theorem 3.4.** *Let  $\Omega$  belong to  $\Xi_d^\infty$ . Then, generically with respect to  $(V, W) \in \mathcal{Z}(\Omega, U)$  the triple  $(\Omega, V, W)$  is fit for control.*

*Proof.* We proved in Proposition 3.2 that each  $\mathcal{R}_k(\Omega)$ , defined in (6), is open and dense in  $\mathcal{V}(\Omega)$ . If  $V$  belongs to  $\mathcal{R}_k(\Omega)$ , then the eigenfunctions  $\phi_1(\Omega, V), \dots, \phi_k(\Omega, V)$  are uniquely defined in  $L^2(\Omega)$  up to sign. It makes sense, therefore, to define

$$\mathcal{U}_k(\Omega, U) = \{(V, W) \in \mathcal{Z}(\Omega, U) \mid V \in \mathcal{R}_k(\Omega), \\ \int_{\Omega} W \phi_{j_1}(\Omega, V) \phi_{j_2}(\Omega, V) \neq 0 \text{ for every } 1 \leq j_1, j_2 \leq k\}.$$

Let  $1 \leq j_1, j_2 \leq k$ . As it follows from the unique continuation property (see [32]), the product  $\phi_{j_1}(\Omega, V) \phi_{j_2}(\Omega, V)$  is a nonzero function on  $\Omega$ . The set of potentials  $W$  belonging to  $\mathcal{W}(\Omega)$  that are not orthogonal to  $\phi_{j_1}(\Omega, V) \phi_{j_2}(\Omega, V)$  is therefore open and dense in  $\mathcal{W}(\Omega)$ . Intersecting all such sets as  $j_1$  and  $j_2$  vary in  $\{1, \dots, k\}$  we obtain again an open and dense subset of  $\mathcal{W}(\Omega)$ . Hence,  $\mathcal{U}_k(\Omega, U)$  is dense in  $\mathcal{Z}(\Omega, U)$ . Its openness, moreover, follows from Proposition 2.9.

The proof is concluded by noticing that  $(\Omega, V, W)$  is fit for control if  $(V, W)$  belongs to

$$(\bigcap_{k \in \mathbf{N}} \mathcal{U}_k(\Omega, U)) \cap \left( \bigcap_{q \in \bigcup_{K \in \mathbf{N}} \mathbf{Q}^K \setminus \{0\}} \{(V, W) \in \mathcal{Z}(\Omega, U) \mid V \in \mathcal{O}_q(\Omega)\} \right),$$

which is a countable intersection of open and dense subsets of  $\mathcal{Z}(\Omega, U)$ .  $\square$

## 4 Generic controllability with respect to one single argument

The following technical result will play a crucial role in the later discussion.

**Lemma 4.1.** *Let  $\Omega$  belong to  $\Xi_d^\infty$  and  $Z$  be a non-constant absolutely continuous function on  $\Omega$ . Then there exist  $\omega \in \Xi_d$  compactly contained in  $\Omega$  with Lipschitz continuous boundary and a reordering  $h : \mathbf{N} \rightarrow \mathbf{N}$  such that  $\sigma(\omega, 0)$  is simple and*

$$\int_{\omega} Z \phi_{h(l)}(\omega, 0) \phi_{h(l+1)}(\omega, 0) \neq 0 \quad (13)$$

for every  $l \in \mathbf{N}$ .

*Proof.* Let  $\bar{x} \in \Omega$  be such that  $\nabla Z(\bar{x})$  exists and is different from zero. Up to a change of coordinates  $\bar{x} = 0$  and each component of  $\nabla Z(0) = (\partial_1 Z(0), \dots, \partial_d Z(0))$  is different from zero.

Take as  $\omega$  an orthotope of the type  $(0, r_1) \times \dots \times (0, r_d)$  such that  $\sigma(\omega, 0)$  is simple. This is true, for instance, if

$$\prod_{\substack{1 \leq i \leq d \\ i \neq j}} r_i^2, \quad j = 1, \dots, d,$$

are  $\mathbf{Q}$ -linearly independent. Let  $r = (r_1, \dots, r_d)$ .

The choice of  $r$  guarantees that  $\sigma(\alpha\omega, 0)$  is simple for every  $\alpha > 0$ . Therefore, the eigenfunctions of  $-\Delta$  on  $\alpha\omega$  are uniquely defined up to sign by

$$\psi_k^\alpha(x_1, \dots, x_d) = \frac{2^{\frac{d}{2}}}{\alpha^{\frac{d}{2}} \sqrt{\prod_{i=1}^d r_i}} \prod_{i=1}^d \sin\left(\frac{k_i x_i \pi}{\alpha r_i}\right)$$

where  $k = (k_1, \dots, k_d)$  belongs to  $\mathbf{N}^d$ .

Denote by  $f_i(k)$  the element of  $\mathbf{N}^d$  obtained from  $k$  by adding 1 to its  $i^{\text{th}}$  component. By construction  $Z(x) = Z(0) + \sum_{j=1}^d x_j \partial_j Z(0) + z(x)$  with  $\lim_{x \rightarrow 0} z(x)/\|x\| = 0$ . Hence,

$$\int_{\alpha\omega} Z(x) \psi_k^\alpha(x) \psi_{f_i(k)}^\alpha(x) dx = \frac{2\partial_i Z(0)}{\alpha r_i} \int_0^{\alpha r_i} x_i \sin\left(\frac{k_i x_i \pi}{\alpha r_i}\right) \sin\left(\frac{(k_i + 1)x_i \pi}{\alpha r_i}\right) dx_i + \varphi_{k,i}(\alpha)$$

with  $|\varphi_{k,i}(\alpha)| \leq \varphi(\alpha)$  and  $\varphi$ , independent of  $k$  and  $i$ , satisfies  $\lim_{\alpha \rightarrow 0} \varphi(\alpha)/\alpha = 0$ . (One can take, for instance,  $\varphi(\alpha) = \sup_{x \in \alpha\omega} |z(x)|$ .)

Notice now that

$$\frac{1}{\alpha^2 r_i^2} \int_0^{\alpha r_i} x_i \sin\left(\frac{k_i x_i \pi}{\alpha r_i}\right) \sin\left(\frac{(k_i + 1)x_i \pi}{\alpha r_i}\right) dx_i = -\frac{4k_i(1 + k_i)}{(1 + 2k_i)^2 \pi^2}$$

and that

$$\lim_{k_i \rightarrow \infty} -\frac{4k_i(1 + k_i)}{(1 + 2k_i)^2 \pi^2} = -\frac{1}{\pi^2}.$$

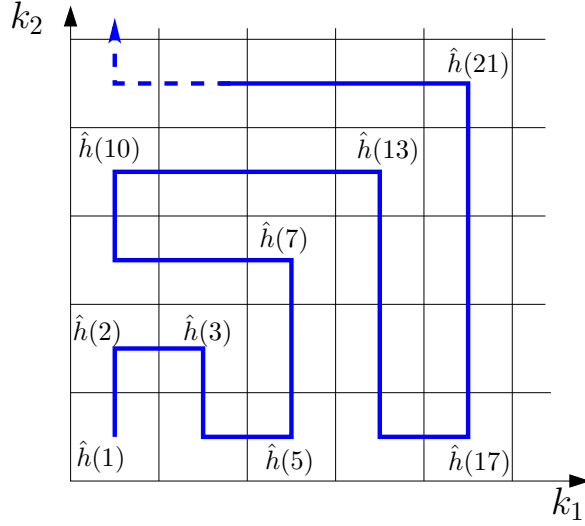


Figure 1: A possible choice of the function  $\hat{h}$  for  $d = 2$ .

Therefore, for  $\alpha$  small enough, we have

$$\int_{\alpha\omega} Z(x) \psi_k^\alpha(x) \psi_{f_i(k)}^\alpha(x) dx \neq 0$$

for every  $k \in \mathbf{N}^d$  and every  $i \in \{1, \dots, d\}$ .

We are left to prove that there exists a bijection  $\hat{h} : \mathbf{N} \rightarrow \mathbf{N}^d$  such that  $\|\hat{h}(j+1) - \hat{h}(j)\| = 1$  for every  $j \in \mathbf{N}$ . This can be interpreted by saying that an infinite-length snake as in [34] can fill  $\mathbf{N}^d$  (see Figure 1).

We claim that the following holds:

For every  $m = (m_1, \dots, m_d) \in \mathbf{N}^d$  such that each  $m_j$  is odd, there exists a bijection

$$\hat{h}_m : \left\{ 1, \dots, \prod_{i=1}^d m_i \right\} \longrightarrow \{1, \dots, m_1\} \times \dots \times \{1, \dots, m_d\}$$

such that  $\|\hat{h}_m(j+1) - \hat{h}_m(j)\| = 1$  for  $j = 1, \dots, \prod_{i=1}^d m_i - 1$ ,  $\hat{h}_m(1) = (1, \dots, 1)$  and  $\hat{h}_m(\prod_{i=1}^d m_i) = m$ . Moreover, if we define  $\bar{m} = (\bar{m}_1, \dots, \bar{m}_d) = (m_1, \dots, m_{p-1}, m_p + 2, m_{p+1}, \dots, m_d)$  for some  $p \in \{1, \dots, d\}$ , then the map  $\hat{h}_m$  can be extended to a bijection

$$\hat{h}_{\bar{m}} : \left\{ 1, \dots, \prod_{i=1}^d \bar{m}_i \right\} \longrightarrow \{1, \dots, \bar{m}_1\} \times \dots \times \{1, \dots, \bar{m}_d\}$$

verifying  $\|\hat{h}_{\bar{m}}(j+1) - \hat{h}_{\bar{m}}(j)\| = 1$  for  $j = 1, \dots, \prod_{i=1}^d \bar{m}_i - 1$ ,  $\hat{h}_{\bar{m}}(1) = (1, \dots, 1)$  and  $\hat{h}_{\bar{m}}(\prod_{i=1}^d \bar{m}_i) = \bar{m}$ .

To prove the existence of such  $\hat{h}_m$  we proceed by induction. For  $d = 1$  the claim is trivial. Let now  $d = \bar{d} > 1$  and assume that the claim is true for  $d = \bar{d} - 1$ . Let  $m = (m_1, \dots, m_{\bar{d}}) \in \mathbf{N}^d$



with  $m_i$  odd for every  $i = 1, \dots, \bar{d}$ . The first part of the claim on  $\hat{h}_m$  is obvious when  $m_{\bar{d}} = 1$  by the inductive assumption. If  $m_{\bar{d}} > 1$  we consider a function  $\hat{h}_{\hat{m}}$  satisfying the first part of the claim with  $\hat{m} = (m_1, \dots, m_{\bar{d}-1})$ . For simplicity denote  $\mu = \prod_{l=1}^{\bar{d}-1} m_l$ . Then the map

$$\hat{h}_m(i) = \begin{cases} (\hat{h}_{\hat{m}}(i - j\mu), j + 1) & \text{for } i = j\mu + 1, \dots, (j + 1)\mu, j \text{ even,} \\ (\hat{h}_{\hat{m}}((j + 1)\mu - i + 1), j + 1) & \text{for } i = j\mu + 1, \dots, (j + 1)\mu, j \text{ odd,} \end{cases}$$

satisfies the required properties. As for the second part of the claim on  $\hat{h}_m$ , let us assume without loss of generality that  $p = \bar{d}$  (if this is not the case it is enough to reorder the indices of  $m_1, \dots, m_{\bar{d}}$ ). Then the map

$$\hat{h}_{\bar{m}}(i) = \begin{cases} \hat{h}_m(i) & \text{for } 1 \leq i \leq \mu m_{\bar{d}} \\ (\hat{h}_{\hat{m}}(\mu(m_{\bar{d}} + 1) - i + 1), m_{\bar{d}} + 1) & \text{for } \mu m_{\bar{d}} + 1 \leq i \leq \mu(m_{\bar{d}} + 1) \\ (\hat{h}_{\hat{m}}(i - \mu(m_{\bar{d}} + 1)), m_{\bar{d}} + 2) & \text{for } \mu(m_{\bar{d}} + 1) + 1 \leq i \leq \mu(m_{\bar{d}} + 2) \end{cases}$$

satisfies the required properties.

To conclude the proof of the existence of  $\hat{h}$  it is enough to consider a sequence of  $d$ -uples  $m(l) = (m_1(l), \dots, m_d(l))$  with positive odd components, such that for every  $l$  there exists  $p$  with  $m_p(l + 1) = m_p(l) + 2$  and  $m_j(l + 1) = m_j(l)$  for  $j \neq p$ , and moreover  $m_j(l)$  goes to infinity as  $l$  goes to infinity for any fixed  $j$ . (Take, for instance,  $p = l \pmod{d} + 1$ .) The map  $\hat{h}$  is then obtained by extending inductively each map  $\hat{h}_{m(l)}$  to a map  $\hat{h}_{m(l+1)}$ .  $\square$

#### 4.1 The triple $(\Omega, V, W)$ is generically fit for control with respect to $V$

Let  $\Omega \in \Xi_d^\infty$  and fix  $W \in \mathcal{W}(\Omega)$ . Let us consider the following subspace of  $\mathcal{V}(\Omega)$

$$\hat{\mathcal{V}}(\Omega, W) = \{V \in \mathcal{V}(\Omega) \mid \operatorname{ess\,sup}_{x \in \Omega} \frac{|W(x)|}{|V(x)| + 1} < +\infty\}.$$

**Theorem 4.2.** *Let  $\Omega$  belong to  $\Xi_d^\infty$  and  $W \in \mathcal{W}(\Omega)$  be non-constant and absolutely continuous. Then, generically with respect to  $V$  in  $\hat{\mathcal{V}}(\Omega, W)$ , the triple  $(\Omega, V, W)$  is fit for control.*

*Proof.* We will denote by  $\mathcal{Q}_n(\Omega, W)$  the set of potentials  $V \in \hat{\mathcal{V}}(\Omega, W)$  such that for every pair of indices  $j, k \in \{1, \dots, n\}$  there exists a finite sequence  $r_1, \dots, r_l \in \mathbf{N}$  such that  $r_1 = j$ ,  $r_l = k$ ,  $\lambda_{r_i}(\Omega, V)$  is simple for every  $i = 1, \dots, l$ , and

$$\int_{\Omega} W \phi_{r_i}(\Omega, V) \phi_{r_{i+1}}(\Omega, V) \neq 0$$

for every  $i = 1, \dots, l - 1$ .

The openness of  $\mathcal{Q}_n(\Omega, W)$  follows from Proposition 2.9.

As for its density, apply Lemma 4.1 with  $W$  playing the role of  $Z$ . Then there exist  $\omega \in \Xi_d$  with Lipschitz boundary and compactly contained in  $\Omega$ , and a reordering  $h$  of  $\mathbf{N}$  such that  $\sigma(\omega, 0)$  is simple and

$$\int_{\omega} W \phi_{h(l)}(\omega, 0) \phi_{h(l+1)}(\omega, 0) \neq 0 \tag{14}$$

for every  $l \in \mathbf{N}$ .

Given  $\bar{V} \in \hat{\mathcal{V}}(\Omega, W)$ , let  $(V_k)_{k \in \mathbf{N}}$  be the sequence in  $\mathcal{V}(\Omega)$  defined by  $V_k = 0$  in  $\omega$  and  $V_k = \bar{V} + k$  in  $\Omega \setminus \omega$ .

Since we know from Lemma 3.3 that  $\|\sqrt{V_k}\phi_j(\Omega, V_k)\|_{L^2(\Omega \setminus \omega)}$  converges to 0 as  $k$  goes to infinity, for every  $j \in \mathbf{N}$ , we have that  $\|\sqrt{|\bar{W}|}\phi_j(\Omega, V_k)\|_{L^2(\Omega \setminus \omega)}$  converges to 0 as  $k$  goes to infinity and, by equation (14), we deduce that there exists  $\bar{k}$  large enough such that  $V_{\bar{k}} \in \mathcal{Q}_n(\Omega, W)$ .

By Proposition 2.12 there exists an analytic function  $\mu \mapsto W_\mu$  from  $[0, 1]$  into  $L^\infty(\Omega)$  such that  $W_0 = 0$ ,  $W_1 = V_{\bar{k}} - \bar{V}$  and the spectrum of  $-\Delta + \bar{V} + W_\mu$  is simple for every  $\mu \in (0, 1)$ . Therefore applying Proposition 2.11 and since  $V_{\bar{k}} = \bar{V} + W_1 \in \mathcal{Q}_n(\Omega, W)$  we get that  $\bar{V} + W_\mu \in \mathcal{Q}_n(\Omega, W)$  for almost every  $\mu \in (0, 1)$ , so that  $\mathcal{Q}_n(\Omega, W)$  is dense in  $\hat{\mathcal{V}}(\Omega, W)$ .

The set  $\cap_{n \in \mathbf{N}} \mathcal{Q}_n(\Omega, W)$  is then residual in  $\hat{\mathcal{V}}(\Omega, W)$ . We claim that if  $V \in \cap_{n \in \mathbf{N}} \mathcal{Q}_n(\Omega, W)$  then there exists a reordering  $\hat{h}$  of  $\mathbf{N}$  such that  $B_n^{\hat{h}}(\Omega, V, W)$  is connected for every  $n \in \mathbf{N}$ . Indeed, let  $\alpha$  be a map from the power set of  $\mathbf{N}$  into itself defined by

$$\alpha(J) = \{m \in \mathbf{N} \setminus J \mid \int_{\Omega} W \phi_n(\Omega, V) \phi_m(\Omega, V) \neq 0 \text{ for some } n \in J\}.$$

Then  $\hat{h}$  can be defined inductively as follows: set  $\hat{h}(1) = 1$  and, for every  $n \in \mathbf{N}$ , let  $\hat{h}(n+1)$  be the smallest element of  $\alpha(\{\hat{h}(1), \dots, \hat{h}(n)\})$ . It is straightforward to check that  $\hat{h}$  is a reordering of  $\mathbf{N}$ .

The triple  $(\Omega, V, W)$  is then fit for control if  $V$  belongs to

$$(\cap_{n \in \mathbf{N}} \mathcal{Q}_n(\Omega, W)) \cap (\cap_{q \in \cup_{K \in \mathbf{N}} \mathbf{Q}^K \setminus \{0\}} \mathcal{O}_q(\Omega))$$

that is the intersection of countably many open and dense subsets of  $\hat{\mathcal{V}}(\Omega, W)$ .  $\square$

**Remark 4.3.** *The proof shows that if  $\sigma(\Omega, V)$  is simple and if, for every pair of indices  $j, k \in \mathbf{N}$ , there exists a finite sequence  $r_1, \dots, r_l \in \mathbf{N}$  such that  $r_1 = j$ ,  $r_l = k$ , and*

$$\int_{\Omega} W \phi_{r_i}(\Omega, V) \phi_{r_{i+1}}(\Omega, V) \neq 0$$

*for every  $i = 1, \dots, l-1$ , then there exists a reordering  $h$  of  $\mathbf{N}$  for which  $B_n^h(\Omega, V, W)$  is connected for every  $n \in \mathbf{N}$ . In particular, if  $\sigma(\Omega, V)$  is simple and for some reordering  $h$  the matrices  $B_n^h(\Omega, V, W)$  are frequently connected (as in the definition of fitness for control), then we can assume without loss of generality that  $B_n^h(\Omega, V, W)$  is connected for every  $n \in \mathbf{N}$ .*

The next corollary follows immediately from Theorem 4.2.

**Corollary 4.4.** *Let  $\Omega \in \Xi_d$  and  $W \in L^\infty(\Omega)$  be non-constant and absolutely continuous. Then, generically with respect to  $V$  in  $L^\infty(\Omega)$ , the triple  $(\Omega, V, W)$  is fit for control.*

In the unbounded case we deduce the following.

**Corollary 4.5.** *Let  $\Omega = \mathbf{R}^d$  and  $W \in \mathcal{W}(\mathbf{R}^d)$  be non-constant and absolutely continuous. Assume that  $U \subset \mathbf{R}$  has nonempty interior. Then, generically with respect to  $V$  in  $\mathcal{V}(\mathbf{R}^d, W, U)$ , the quadruple  $(\mathbf{R}^d, V, W, U)$  is effective.*

*Proof.* Let  $u$  belong to the interior of  $U$ . Assume in particular  $[u - \delta, u + \delta] \subset U$ . Then, from the definition of  $\mathcal{V}(\mathbf{R}^d, W, U)$  we have that  $V + uW + \delta W$  and  $V + uW - \delta W$  are both positive outside a bounded subset  $\Omega_0$  of  $\mathbf{R}^d$ . In particular  $|W| \leq \frac{1}{\delta}|V + uW|$  outside  $\Omega_0$ , while  $W$  is bounded on  $\Omega_0$ . Therefore  $V + uW \in \hat{\mathcal{V}}(\mathbf{R}^d, W)$  and applying Theorem 4.2, we have that the triple  $(\mathbf{R}^d, V + uW, W)$  is fit for control, generically with respect to  $V \in \mathcal{V}(\mathbf{R}^d, W, U)$ .  $\square$

## 4.2 The quadruple $(\Omega, V, W, U)$ is generically effective with respect to $W$

We prove in this section that for a fixed potential  $V$ , generically with respect to  $W \in \mathcal{W}(\Omega, V, U)$ , the quadruple  $(\Omega, V, W, U)$  is effective. Notice that  $(\Omega, V, W)$  cannot be fit for control if the spectrum of  $-\Delta + V$  is resonant, independently of  $W$ . In this regard the result is necessarily weaker than Theorems 3.4 and 4.2, where the genericity of the fitness for control was proved.

**Proposition 4.6.** *Let  $\Omega$  belong to  $\Xi_d^\infty$  and  $V \in \mathcal{V}(\Omega)$  be an absolutely continuous function on  $\Omega$ . Assume that  $U$  has nonempty interior. Then, generically with respect to  $W \in \mathcal{W}(\Omega, V, U)$ , the quadruple  $(\Omega, V, W, U)$  is effective.*

*Proof.* Fix  $u \neq 0$  in the interior of  $U$ . Notice that  $V + u\mathcal{W}(\Omega, V, U)$  is an open subset of  $\mathcal{V}(\Omega)$  diffeomorphic to  $\mathcal{W}(\Omega, V, U)$ . In particular, for every  $K \in \mathbf{N}$  and  $q \in \mathbf{Q}^K \setminus \{0\}$ , the set  $\{W \in \mathcal{W}(\Omega, V, U) \mid V + uW \in \mathcal{O}_q(\Omega)\}$  is open and dense in  $\mathcal{W}(\Omega, V, U)$ .

For every  $W \in \mathcal{W}(\Omega, V, U)$  let  $\mathcal{Q}_n(\Omega, W)$  be defined as in the previous section. As proved in Corollary 4.5, for every  $W \in \mathcal{W}(\Omega, V, U)$  one has  $V + uW \in \hat{\mathcal{V}}(\Omega, W)$ . We prove the proposition by showing that for every  $n \in \mathbf{N}$ , for each  $W$  in a open and dense subset of  $\mathcal{W}(\Omega, V, U)$  (depending on  $n$ ),  $V + uW$  belongs to  $\mathcal{Q}_n(\Omega, W)$ .

Define

$$\mathcal{P}_n = \{W \in \mathcal{W}(\Omega, V, U) \mid V + uW \in \mathcal{Q}_n(\Omega, W)\}.$$

Because of Remark 4.3 it is enough to prove that each  $\mathcal{P}_n$  is open and dense. Since

$$W \mapsto \int_{\Omega} W \phi_j(\Omega, V + uW) \phi_k(\Omega, V + uW).$$

is continuous on  $\{W \in \mathcal{W}(\Omega, V, U) \mid \lambda_j(\Omega, V + uW), \lambda_k(\Omega, V + uW) \text{ are simple}\}$  for every  $j, k \in \mathbf{N}$  (Proposition 2.9), we deduce that  $\mathcal{P}_n$  is open.

Fix  $\overline{W} \in \mathcal{W}(\Omega, V, U)$ . We are left to prove that  $\overline{W}$  belongs to the closure of  $\mathcal{P}_n$ .

Consider first the case in which  $V$  is constant. In particular,  $\Omega \in \Xi_d$ ,  $\mathcal{W}(\Omega, V, U) = V + u\mathcal{W}(\Omega, V, U) = L^\infty(\Omega)$ , and

$$\int_{\Omega} W \phi_j(\Omega, V + uW) \phi_k(\Omega, V + uW) = \int_{\Omega} W \phi_j(\Omega, uW) \phi_k(\Omega, uW). \quad (15)$$

Fix  $\omega \in \Xi_d$  compactly contained in  $\Omega$ , whose boundary is Lipschitz continuous and such that the spectrum  $\sigma(\omega, 0)$  is simple. For instance,  $\omega$  can be taken as an orthotope whose side's lengths are non-resonant. (The simplicity of the spectrum of the Laplace-Dirichlet operator on  $\omega$  is actually generic among sufficiently smooth domains, as proved in [21, 36].)

Let  $z \in L^\infty(\omega)$  be non-orthogonal in  $L^2(\omega)$  to  $\phi_j(\omega, 0)\phi_k(\omega, 0)$  for every  $j, k \in \mathbf{N}$ . (Such  $z$  exists because each  $\phi_j(\omega, 0)\phi_k(\omega, 0)$  is not identically equal to zero and because  $L^\infty(\omega)$  is a Baire space.) Then, for every  $j, k \in \mathbf{N}$ , the derivative of

$$\varepsilon \mapsto \int_{\omega} \varepsilon z \phi_j(\omega, \varepsilon z) \phi_k(\omega, \varepsilon z)$$

at  $\varepsilon = 0$  is equal to

$$\int_{\omega} z \phi_j(\omega, 0) \phi_k(\omega, 0) \neq 0.$$

By Theorem 2.10, there exists  $\bar{\varepsilon} \in \mathbf{R}$  such that the spectrum  $\sigma(\omega, \bar{\varepsilon} z)$  is simple and

$$\int_{\omega} \bar{\varepsilon} z \phi_j(\omega, \bar{\varepsilon} z) \phi_k(\omega, \bar{\varepsilon} z) \neq 0$$

for every  $j, k \in \mathbf{N}$ .

Let  $(W_l)_{l \in \mathbf{N}}$  be a sequence in  $L^\infty(\Omega)$  such that  $W_l - \bar{W} \in L^\infty(\Omega)$ ,  $\lim_{l \rightarrow \infty} W_l|_{\omega} = (\bar{\varepsilon}/u)z$  in  $L^2(\omega)$  and  $\lim_{l \rightarrow \infty} \inf_{\Omega \setminus \omega} W_l = +\infty$ . By Lemma 3.3 we deduce that there exists  $\bar{l}$  large enough such that

$$\int_{\Omega} W_{\bar{l}} \phi_j(\Omega, u W_{\bar{l}}) \phi_k(\Omega, u W_{\bar{l}}) \neq 0 \quad \text{for } j, k = 1, \dots, n.$$

By Proposition 2.12 we can consider an analytic curve  $\mu \mapsto \hat{W}_\mu$  in  $L^\infty(\Omega)$  for  $\mu \in [0, 1]$  such that  $\hat{W}_0 = \bar{W}$ ,  $\hat{W}_1 = W_{\bar{l}}$  and the spectrum of  $-\Delta + u \hat{W}_\mu$  is simple for every  $\mu \in (0, 1)$ , and we have

$$\int_{\Omega} \hat{W}_\mu \phi_j(\Omega, V + u \hat{W}_\mu) \phi_k(\Omega, V + u \hat{W}_\mu) = \int_{\Omega} \hat{W}_\mu \phi_j(\Omega, u \hat{W}_\mu) \phi_k(\Omega, u \hat{W}_\mu) \neq 0$$

for almost every  $\mu \in (0, 1)$  and in particular for some  $\mu$  arbitrarily small, implying that  $\bar{W}$  belongs to the closure of  $\mathcal{P}_n$ .

Let now  $V$  be non-constant. Let  $\omega \subset \Omega$  and  $h$  be as in the statement of Lemma 4.1 with  $V$  playing the role of  $Z$ .

Take a sequence  $(W_k)_{k \in \mathbf{N}}$  in  $\mathcal{W}(\Omega, V, U)$  such that  $W_k - \bar{W}$  belongs to  $L^\infty(\Omega)$  for every  $k$  and

$$\lim_{k \rightarrow +\infty} \|V + u W_k\|_{L^\infty(\omega)} = 0, \quad \lim_{k \rightarrow +\infty} \inf_{\Omega \setminus \omega} (u W_k) = +\infty.$$

According to Lemma 3.3,

$$\lim_{k \rightarrow +\infty} \phi_m(\Omega, V + u W_k) = \phi_m(\omega, 0) \quad \text{and} \quad \lim_{k \rightarrow +\infty} \sqrt{V + u W_k} \phi_m(\Omega, V + u W_k) = 0$$

in  $L^2(\Omega, \mathbf{C})$  for every  $m \in \mathbf{N}$ , where  $\phi_m(\omega, 0)$  is identified with its extension by zero on  $\Omega \setminus \omega$ . In particular, we have that  $\sqrt{V} \phi_m(\Omega, V + u W_k)$  converges in  $L^2(\Omega, \mathbf{C})$  as  $k$  tends to infinity to the extension by zero of  $\sqrt{V} \phi_m(\omega, 0)$  on  $\Omega \setminus \omega$ . Hence,

$$\lim_{k \rightarrow +\infty} \int_{\Omega} W_k \phi_{h(l)}(\Omega, V + u W_k) \phi_{h(l+1)}(\Omega, V + u W_k) = -\frac{1}{u} \int_{\omega} V \phi_{h(l)}(\omega, 0) \phi_{h(l+1)}(\omega, 0) \neq 0,$$

for every  $l \in \mathbf{N}$ . For a fixed  $n \in \mathbf{N}$ , we can choose  $\bar{k}$  large enough so that

$$\int_{\Omega} W_{\bar{k}} \phi_{h(l)}(\Omega, V + u W_{\bar{k}}) \phi_{h(l+1)}(\Omega, V + u W_{\bar{k}}) \neq 0,$$

for  $l$  large enough, in order to guarantee that  $W_{\bar{k}} \in \mathcal{P}_n$ . By Proposition 2.12 there exists an analytic path  $\mu \mapsto \hat{W}_\mu$  from  $[0, 1]$  into  $L^\infty(\Omega)$  such that  $\hat{W}_0 = 0$ ,  $\hat{W}_1 = W_{\bar{k}} - \overline{W}$  and the spectrum of  $-\Delta + V + u\overline{W} + u\hat{W}_\mu$  is simple for every  $\mu \in (0, 1)$ . Therefore, by analyticity of the eigenfunctions and by applying Proposition 2.11, we get that

$$\int_{\Omega} (\overline{W} + \hat{W}_\mu) \phi_{h(l)}(\Omega, V + u\overline{W} + u\hat{W}_\mu) \phi_{h(l+1)}(\Omega, V + u\overline{W} + u\hat{W}_\mu) \neq 0$$

for almost every  $\mu \in (0, 1)$ .

Hence,  $\overline{W}$  belongs to the closure of  $\mathcal{P}_n$ . □

**Remark 4.7.** *It seems possible to adapt the arguments presented above and in the previous section to the conditions ensuring approximate controllability in the recent work by Nersesyan [25]: namely, that there exists a reordering  $h$  such that  $\lambda_{h(1)}(\Omega, V) - \lambda_{h(j)}(\Omega, V) \neq \lambda_{h(p)}(\Omega, V) - \lambda_{h(q)}(\Omega, V)$  for all  $j, p, q \in \mathbf{N}$  such that  $\{1, j\} \neq \{p, q\}$  and  $j \neq 1$ , and that the first line of  $B_n^h(\Omega, V, W)$  is made of non-zero elements for every  $n \in \mathbf{N}$  ([25] also requires that  $\Omega$  is bounded, with smooth boundary and that  $V, W$  are smooth up to the boundary). In order to do so, a counterpart of Lemma 4.1 should be proved, replacing (13) by*

$$\int_{\omega} Z \phi_{h(1)}(\omega, 0) \phi_{h(l)}(\omega, 0) \neq 0, \quad \text{for every } l \in \mathbf{N}.$$

*This is done in [11] for the case  $d = 2$  (just replace  $\mu$  by  $(Z, 0)$  in Proposition 2.8).*

## 5 Conclusion

In this paper we proved that once  $(\Omega, V)$  or  $(\Omega, W)$  is fixed (with  $W$  non-constant), the bilinear Schrödinger equation on  $\Omega$  having  $V$  as uncontrolled and  $W$  as controlled potential is generically approximately controllable in the sense of the density matrices with respect to the other element of the triple  $(\Omega, V, W)$ .

A natural question is whether a similar property holds with respect to the dependence on  $\Omega$ . It makes sense to conjecture that it does but the proof of this fact seems hard to obtain through the techniques used here. Fix  $V$  and  $W$  absolutely continuous on  $\mathbf{R}^d$  with  $W$  nowhere locally constant. Let  $m \in \mathbf{N}$  and  $\Omega$  belong to the space of bounded  $\mathcal{C}^m$  domains endowed with the  $\mathcal{C}^m$  topology (this space is Baire as proved in [20]). One important remark is that the dependence of  $\lambda_k(\Omega, V)$  on  $\Omega$  is not necessarily analytic, as it would be the case if  $V$  was analytic. (A genericity non-resonance result for the spectrum in the case  $V = 0$ , for instance, has been proved along these lines in [26].) Similarly, the quantities  $\int_{\Omega} W \phi_k(\Omega, V) \phi_j(\Omega, V)$  do not in general vary analytically with respect to  $\Omega$ . Hence, the pattern of the proofs seen in the previous sections could not be followed. A partial result going in the right direction can be found in [11], where the authors prove that for  $V = 0$  and  $W \in \mathcal{C}^1(\mathbf{R}^2, \mathbf{R})$  nowhere-constant, for a generic  $\mathcal{C}^3$  domain  $\Omega \subset \mathbf{R}^2$  one has  $\int_{\Omega} W \phi_1(\Omega, 0) \phi_j(\Omega, 0) \neq 0$  for every  $j \in \mathbf{N}$ . The proof of this fact in [11] is very technical and ingenious. Its extension to general uncontrolled potentials and to the case  $d > 2$  looks an extremely hard task.

## 6 Acknowledgments

The authors are grateful to Ugo Boscain, Thomas Chambrion, Yacine Chitour, Antoine Henrot and Karim Ramdani for helpful discussions.

## References

- [1] R. Adami and U. Boscain. Controllability of the Schrödinger equation via intersection of eigenvalues. In *Proceedings of the 44th IEEE Conference on Decision and Control, December 12-15*, pages 1080–1085, 2005.
- [2] S. Agmon. Bounds on exponential decay of eigenfunctions of Schrödinger operators. In *Schrödinger operators (Como, 1984)*, volume 1159 of *Lecture Notes in Math.*, pages 1–38. Springer, Berlin, 1985.
- [3] A. A. Agrachev and A. V. Sarychev. Controllability of 2D Euler and Navier-Stokes equations by degenerate forcing. *Comm. Math. Phys.*, 265(3):673–697, 2006.
- [4] J. H. Albert. Genericity of simple eigenvalues for elliptic PDE’s. *Proc. Amer. Math. Soc.*, 48:413–418, 1975.
- [5] F. Albertini and D. D’Alessandro. Notions of controllability for bilinear multilevel quantum systems. *IEEE Trans. Automat. Control*, 48(8):1399–1403, 2003.
- [6] G. Allaire and R. Orive. Homogenization of periodic non self-adjoint problems with large drift and potential. *ESAIM Control Optim. Calc. Var.*, 13(4):735–749, 2007.
- [7] C. Altafini. Controllability properties for finite dimensional quantum Markovian master equations. *J. Math. Phys.*, 44(6):2357–2372, 2003.
- [8] P. M. Anselone. *Collectively compact operator approximation theory and applications to integral equations*. Prentice-Hall Inc., Englewood Cliffs, N. J., 1971.
- [9] J. M. Ball, J. E. Marsden, and M. Slemrod. Controllability for distributed bilinear systems. *SIAM J. Control Optim.*, 20(4):575–597, 1982.
- [10] K. Beauchard. Local controllability of a 1-D Schrödinger equation. *J. Math. Pures Appl.* (9), 84(7):851–956, 2005.
- [11] K. Beauchard, Y. Chitour, D. Kateb, and R. Long. Spectral controllability for 2D and 3D linear Schrödinger equations. *J. Funct. Anal.*, 256(12):3916–3976, 2009.
- [12] K. Beauchard and J.-M. Coron. Controllability of a quantum particle in a moving potential well. *J. Funct. Anal.*, 232(2):328–389, 2006.
- [13] M. S. Byrd and N. Khaneja. Characterization of the positivity of the density matrix in terms of the coherence vector representation. *Phys. Rev. A* (3), 68(6):062322, 13, 2003.
- [14] T. Chambrion. Approximate tracking for a system of Schrödinger equations. *Preprint*, 2009.



- [15] T. Chambrion, P. Mason, M. Sigalotti, and U. Boscain. Controllability of the discrete-spectrum Schrödinger equation driven by an external field. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(1):329–349, 2009.
- [16] K. Friedrichs. Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren. *Math. Ann.*, 109(1):465–487, 685–713, 1934.
- [17] A. Henrot and M. Pierre. *Variation et optimisation de formes. Une analyse géométrique*, volume 48 of *Mathématiques et Applications*. Springer-Verlag, Berlin, 2005.
- [18] T. Kato. *Perturbation theory for linear operators*. Die Grundlehren der mathematischen Wissenschaften, Band 132. Springer-Verlag New York, Inc., New York, 1966.
- [19] E. H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1997.
- [20] A. M. Micheletti. Metrica per famiglie di domini limitati e proprietà generiche degli autovalori. *Ann. Scuola Norm. Sup. Pisa (3)*, 26:683–694, 1972.
- [21] A. M. Micheletti. Perturbazione dello spettro dell’operatore di Laplace, in relazione ad una variazione del campo. *Ann. Scuola Norm. Sup. Pisa (3)*, 26:151–169, 1972.
- [22] M. Mirrahimi. Lyapunov control of a quantum particle in a decaying potential. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1743–1765, 2009.
- [23] M. Mirrahimi and P. Rouchon. Controllability of quantum harmonic oscillators. *IEEE Trans. Automat. Control*, 49(5):745–747, 2004.
- [24] V. Nersesyan. Global approximate controllability for Schrödinger equation in higher Sobolev norms and applications. *Preprint*, 2009.
- [25] V. Nersesyan. Growth of Sobolev norms and controllability of the Schrödinger equation. *Comm. Math. Phys.*, 290(1):371–387, 2009.
- [26] Y. Privat and M. Sigalotti. The squares of the Laplacian-Dirichlet eigenfunctions are generically linearly independent. *ESAIM Control Optim. Calc. Var.*, to appear.
- [27] M. Reed and B. Simon. *Methods of modern mathematical physics. IV. Analysis of operators*. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1978.
- [28] M. Reed and B. Simon. *Methods of modern mathematical physics. I*. Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, second edition, 1980. Functional analysis.
- [29] F. Rellich. *Perturbation theory of eigenvalue problems*. Assisted by J. Berkowitz. With a preface by Jacob T. Schwartz. Gordon and Breach Science Publishers, New York, 1969.
- [30] M. Reuquardt and A. Schlömerkemper. Perturbation theory of Schrödinger operators in infinitely many coupling parameters. *J. Phys. A*, 32(43):7523–7541, 1999.



- [31] S. S. Rodrigues. Navier-Stokes equation on the rectangle controllability by means of low mode forcing. *J. Dyn. Control Syst.*, 12(4):517–562, 2006.
- [32] B. Simon. Schrödinger semigroups. *Bull. Amer. Math. Soc. (N.S.)*, 7(3):447–526, 1982.
- [33] M. Teytel. How rare are multiple eigenvalues? *Comm. Pure Appl. Math.*, 52(8):917–934, 1999.
- [34] P. Trefonas. Worm. *CLOAD Magazine*, 13, 1979.
- [35] G. Turinici. On the controllability of bilinear quantum systems. In M. Defranceschi and C. Le Bris, editors, *Mathematical models and methods for ab initio Quantum Chemistry*, volume 74 of *Lecture Notes in Chemistry*. Springer, 2000.
- [36] K. Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98(4):1059–1078, 1976.